

# Reachability and recurrence in a modular generalization of annihilating random walks (and lights-out games) on hypergraphs

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## Abstract

We study a dynamical system motivated by our earlier work [1] on the statistical physics of social balance on graphs [2, 3], that can be viewed as a generalization of annihilating walks along two directions: first, the interaction topology is a hypergraph; second, the “number of particles” at a vertex of the hypergraph is an element of a finite field  $\mathbf{Z}_p$  of integers modulo  $p$ ,  $p \geq 3$ . Equivalently, particles move on a hypergraph, with a moving particle at a vertex being replaced by one indistinguishable copy at each neighbor in a given hyperedge; particles at a vertex collectively annihilate when their number reaches  $p$ . The system we study can also be regarded as a natural generalization of certain lights-out games [4] to finite fields and hypergraph topologies.

Our result shows that under a liberal sufficient condition on the nature of the interaction hypergraph there exists a polynomial time algorithm (based on linear algebra over  $\mathbf{Z}_p$ ) for deciding reachability and recurrence of this dynamical system. Interestingly, we provide a counterexample that shows that this connection does *not* extend to all graphs.

# 1 Introduction

Coalescing and annihilating random walks are important tools in the theory of interacting particle systems [5] (where they serve as duals of the fundamental *voter* and *antivoter* model). They are also useful in the theory of discrete Markov chains [6], particularly related to *perfect sampling* via coupling from the past [7].

A model from the Statistical Physics of social balance [2, 3] led us in [1] to considering an extension of annihilating random walks to hypergraphs.

In this note we study the  $\mathbf{Z}_p$ -version of the system from [1]. Specifically, we consider

**Definition 1** *Let  $p \geq 2$  be an integer. A  $\mathbf{Z}_p$ -annihilating walk on a hypergraph  $G$  is defined as follows: each node  $v$  of  $G$  is initially endowed with a number  $b_v \in \mathbf{Z}_p$  (interpreted as number of particles).*

*The allowed moves are specified as follows: choose a node  $v$  such that  $b_v \neq 0$  and a hyperedge  $e$  containing  $v$ . Change the state of  $b_v$  to  $b_v - 1$ . Also change the state of every node  $w \neq v, w \in e$  to  $b_w + 1$ .*

The dynamics from Definition 1 has a very intuitive description: a number of indistinguishable particles are initially placed at the vertices of  $G$ , each vertex holding from 0 to  $p - 1$  particles. At each step we choose a vertex  $v$  containing at least one particle and a hyperedge containing  $v$ . We delete one particle at  $v$  and add one particle at every vertex  $w \neq v \in e$ . If the number of particles at some  $w$  reaches  $p$ , these  $p$  particles are removed from  $w$  (they "collectively annihilate").

We are mainly interested in the complexity of the following two problems:

**Definition 2 (REACHABILITY)** *Given hypergraph  $G = (E, V)$  and states  $w_1, w_2 \in \mathbf{Z}_p^V$ , decide whether  $w_2$  is reachable from  $w_1$ .*

**Definition 3 (RECURRENCE)** *Given hypergraph  $G = (E, V)$  and states  $w_1, w_2 \in \mathbf{Z}_p^V$ , decide whether  $w_2$  is reachable from any state  $w_3 \in \mathbf{Z}_p^V$  reachable from  $w_1$ .*

Of course, reachability and recurrence are fundamental prerequisites for studying the *random* version of this dynamical system as a finite-state Markov chains, the problem that was the original motivation of our research.

There are simple algorithms that put the complexity of these two problems above in the complexity classes PSPACE and EXPSPACE, respectively: for REACHABILITY we simply consider reachability in the (exponentially large) state space directed graph  $S$  with vertex set  $\mathbf{Z}_p^V$ . For RECURRENCE we combine enumeration of all vertices  $w_3$  reachable from  $w_1$  (via breadth first search) with testing reachability of  $w_2$  from  $w_3$ .

The main purpose of this paper is to show that under a quite liberal sufficient condition, reachability and recurrence questions for  $\mathbf{Z}_p$ -annihilating walks on hypergraphs can be decided in polynomial time (actually they belong to the

apparently weaker class  $Mod_p\text{-L}$  [8], but we won't discuss this issue here any further), by solving a certain system of linear equations over  $Z_p$ .

For simplicity of modular arithmetic, in this paper we will assume that  $p \geq 3$  is a prime (so that  $\mathbf{Z}_p$  is a field). This is most likely not an essential assumption, but it will make some of the tricks employed (e.g. Observation 2 below) easier.

## 2 Related Work

After completing our paper [1] we found out that the dynamics we study there is naturally related to a classical problem in the area of combinatorial games. Specifically, the dynamics studied in [1] is a generalization to hypergraphs of a variant of the *lights out* ( $\sigma$ )-game [4], a problem that has seen significant investigation. The version we considered in [1] is the apparently more constrained *lit-only*  $\sigma^+$ -game:

**Definition 4** *Let  $G = (V, E)$  be a finite graph. Each vertex  $v \in V$  has a lightbulb (that is either "on" or "off") and a light switch. In the lights out ( $\sigma$ )-game pressing the light switch at any given vertex  $v$  changes the state of the lightbulbs at all neighbors of  $v$ . In the  $\sigma^+$ -game the action also changes the state of the lightbulb at  $v$ .*

*The lit-only versions of the  $\sigma$  and  $\sigma^+$  games only allow toggling switches of lit vertices.*

Sutner [4] showed that the all zeros state is reachable from the all-ones state in the  $\sigma^+$ -game. This was generalized to Scherphuis [9] to the lit-only  $\sigma^+$ -game. A recent result ([10] Theorem 3) significantly overlaps with our result in [1], essentially showing that the lit-restriction does not make a difference for reachability on hypergraphs that arise as so-called *neighborhood hypergraphs* [11] of a given graph; this result is incomparable to ours, as it does not require, as we do, that the degree of each hyperedge to be at least three; on the other hand we do not restrict ourselves to neighborhood hypergraphs<sup>1</sup>.

Lights out games were considered for finite fields  $\mathbf{Z}_p$ ,  $p \neq 2$  as well, e.g. in [12]. Our framework differs from the one in that paper in several important ways: first we consider the  $\sigma^+$ -game (rather than the  $\sigma$ -game). Second our definition differs slightly in the specification of the dynamics, as the value of the scheduled vertex *decreases*, rather than increases, by one (as it does in [12]). The motivation for this variation is our desired connection with the theory of *interacting particle systems* [5], particularly with the definition of coalescing/annihilating random walks.

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<sup>1</sup>At a first glance it would seem that our result is inconsistent with that in [10], in that it omits an important condition in the statement of Theorem 3 in [10]. This required condition is that the final state should not be "all ones" on any connected component. The difference in statements is, however, only apparent: a simple argument shows that if the final state  $w_2$  is the "all ones" state then the only state  $w_1$  allowed by our result as a preimage of  $w_2$  is  $w_1 = w_2$ .

The further connections with this latter theory are also worth mentioning: threshold coalescing and annihilating random walks, where several particles have to be present at a site for interaction with the new particle to occur, have previously been studied (e.g. [13]) in the interacting particle systems literature. Compared to this work our results differ in an important respect: instead of working on a lattice like  $\mathbf{Z}^d$  our result considers the case of a finite hypergraph. Remarkably few results in this area (e.g. [14], [15], see also [6] Chapter 14) consider the case of a finite graph topology, much less that of a finite hypergraph.

Finally, we briefly discuss the connections between the dynamical model studied in this paper and the Statistical Physics of social dynamics [16]. As stated, a model inspired by the sociological theory of *social balance* [17] that originated in the Statistical Physics literature [2, 3], was the original motivation for our work [1]. We do not see how to sensibly extend the model in [2, 3] so that it corresponds to our generalization of annihilating random walks. On the other hand such walks correspond via *duality* (see [6] Chapter 14 and [18]) to a fundamental model of opinion dynamics, the *antivoter model*. "Cyclic" extensions of antivoter models have been investigated as well (e.g. [19, 20]), and we can define such a "cyclic" extension that corresponds via duality to our  $\mathbf{Z}_p$ -generalization of annihilating random walks. Details (and a more complete study of our system as a Markov chain) are left for future work.

### 3 The Main Result

Let  $Z_1$  and  $Z_2 \in \mathbf{Z}_p^V$  be states of the system such that  $Z_2$  is reachable from  $Z_1$ . Define variables  $X_{e,v}$  denoting the number of times (modulo  $p$ ) that vertex  $v$  and hyperedge  $e$  are chosen in the process from Definition 1. The effect of scheduling pair  $(v, e)$ , given current configuration  $Z$ , is to modify the value of  $Z(v)$  by  $-1$  and of all  $Z(w)$  by  $+1 \pmod p$ . Hence:

$$\sum_{v \in e} X_{e,v} - \sum_{\substack{v \neq w, \\ v, w \in e}} X_{e,w} = w_1(v) - w_2(v) \pmod p \quad (1)$$

We will denote by  $H(w_1, w_2, G)$  the system of equations (1).

Does the converse hold? I.e. is the solvability of system  $H(w_1, w_2, G)$  sufficient for the state  $w_2$  to be reachable from  $w_1$ ?

**Observation 1** *In general the answer to the previous question is negative: Figure 1 provides a counterexample: from state  $w_1 = [1; 1; 1]$  one cannot reach state  $w_2 = [2; 2; 2]$ , even though the system has a solution in  $\mathbf{Z}_3$ . Indeed, the only other configurations reachable from  $[1; 1; 1]$  are easily seen to be  $[0; 1; 2]$  and its permutations, as well as  $[0; 0; 0]$ . Alternatively, state  $[2; 2; 2]$  is a garden-of-Eden state, since no edge can be the last one to be scheduled and produce configuration  $[2; 2; 2]$  on its vertices; indeed, the scheduled vertex would have had zero particles to start with, making the move illegal.*

Nevertheless, in some conditions a converse does actually hold:

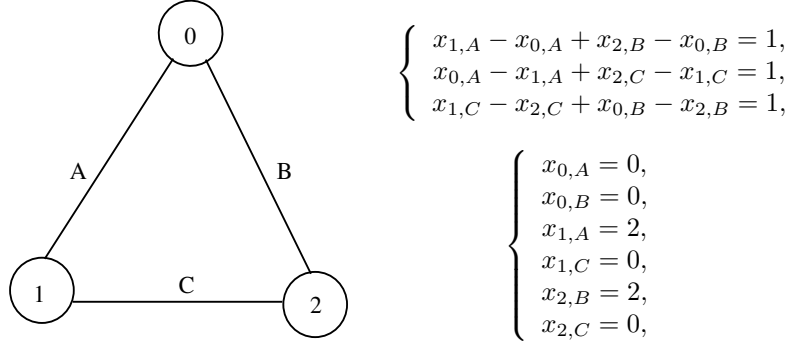


Figure 1: (a). The counterexample  $G$  ( $p=3$ ) (b). The system and its solution

**Definition 5** A hypergraph  $G$  is good if:

- $G$  is connected.
- For every hyperedge  $e \in E(G)$ ,  $|e| \geq 3$ .
- For every two hyperedges  $e_1 \neq e_2 \in E(G)$ ,  $|e_1 \cap e_2| \leq 1$ . In particular,  $G$  is simple, i.e. for no two hyperedges  $e_1, e_2$  it holds that  $e_1 \subseteq e_2$ <sup>2</sup>.

We have to impose conditions in Definition 5 to obtain

**Theorem 1** Let  $G$  be a good hypergraph. Let  $w_1$  be an initial configuration that is not identical to the "all zeros" configuration  $\mathbf{0}$ , and let  $w_2$  be a final configuration.

Then  $w_2$  is reachable from state  $w_1$  if and only if system  $H(w_1, w_2, G)$  has a solution in  $\mathbf{Z}_p$ .

**Proof 1** We will need the following definitions:

**Definition 6** When system  $H(w_1, w_2, G)$  is solvable we define the norm of the system  $H(w_1, w_2, G)$  as the quantity

$$|H(w_1, w_2, G)| = x_1 + x_2 + \dots + x_p,$$

where  $x$  ranges over all solutions of the system and the  $x_i$ 's and the sum are taken in  $\mathbf{Z}$ , rather than  $\mathbf{Z}_p$ .

**Definition 7** Let  $G = (V, E)$  be a hypergraph,  $l \in E$  be a hyperedge in  $G$ , and  $v \in l$  a vertex. We define state vector  $a_{v,l} \in \mathbf{Z}_p^V$  by

$$a_{v,l}(z) = \begin{cases} +1 & , \text{ if } z = v, \\ -1 & , \text{ if } z \neq v, z \in l, \\ 0 & , \text{ otherwise.} \end{cases}$$

<sup>2</sup>In [1] this condition was implicitly assumed, being true for hypergraphs arising via so-called *triadic duality* from the original problem about social balance.

**Definition 8** Let  $G = (V, E)$  be a hypergraph,  $l \in E$  be a hyperedge in  $G$ ,  $w \in \mathbf{Z}_p^V$  be a state and  $a \in \mathbf{Z}_p^V$ . We denote by  $w^{[a,l]}$  the following state:

$$w^{[a,l]}(v) = \begin{cases} w(v) & , \text{ if } v \notin l, \\ w(v) + a(v) & , \text{ otherwise.} \end{cases}$$

Also, with the conventions in the previous definition, we will write  $w^{[v,l]}$  instead of  $w^{[a_v,l]}$  and, for  $k \geq 1$ ,  $w^{[k,v,l]}$  instead of  $w^{[k \cdot a_v,l]}$ . Vector  $w^{[k,v,l]}$  can be interpreted as applying  $k$  moves at vertex  $v$  on edge  $l$ .

**Definition 9** A pair of vertices  $(v_1, v_2)$  is good in state  $w \in \mathbf{Z}_p^V$  if  $(w(v_1), w(v_2)) \notin \{(0, 0), (p-1, p-1)\}$ .

We first make the following simple

**Observation 2** Let  $C$  be a configuration on hypergraph  $G$  and  $v_1 \neq v_2$  two vertices of  $G$  in the same hyperedge  $e$  such that pair  $(v_1, v_2)$  is good in  $C$ . Then one can change configuration  $C$  into configuration  $D$  that has the same number of particles at  $v_1, v_2$  but the number of particles at any other vertex  $v$  of  $e$  increases by one (mod  $p$ ). The move only involves edge  $e$  and some of its vertices. A similar statement holds for decreasing labels by one (mod  $p$ ), instead of increasing them.

**Proof 2** If  $\text{label}(v_1) \neq 0$  and  $\text{label}(v_2) \neq p-1$  first make a move at vertex  $v_1$  then make a move at vertex  $v_2$ . The number of particles at  $v_1, v_2$  stays the same, whereas it increases by two (mod  $p$ ) at any other vertex. Since  $p \geq 3$  is prime,  $p$  is relatively prime to 2. We repeat this process  $k$  times, where  $k$  is chosen such that  $2k = 1 \pmod{p}$ . If  $\text{label}(v_2) = p-1$  then  $\text{label}(v_1) \neq p-1 \pmod{p}$ , so we may repeat the above scheme with moves first made at  $v_2$  then at  $v_1$ .

The proof for the second case is identical, with  $2k = -1 \pmod{p}$ .  $\square$

We prove Theorem 1 by induction on  $m$ , the number of hyperedges of  $G$ .

- **Case  $m = 1$ :** Suppose system  $H(w_1, w_2, G)$  has a solution. Since  $G$  contains a single edge  $e$ ,  $w_2(v) = w_1(v)$  for all vertices  $v \notin e$  (otherwise the system would contain an equation  $0 = \lambda$ , with  $0 \neq \lambda = w_2(v) - w_1(v) \in \mathbf{Z}_p \setminus \{0\}$ ). Thus we can assume that  $G$  consists of exactly those vertices  $v_1, v_2, \dots, v_k$  connected by edge  $e$ . Denote  $\bar{w}$  the vector  $w_2 - w_1$  and, for simplicity, let  $w_1, w_2, \dots, w_k$  be shorthands for  $\bar{w}[v_1], \bar{w}[v_2], \dots, \bar{w}[v_k]$ . Similarly, let  $w_{a,i}$  stand for  $w_a[v_i]$ , where  $i = 1, \dots, k$ ,  $a = 1, 2$ . Also define  $w = w_1 + w_2 + \dots + w_k$ . System  $H(w_1, w_2, G)$  reads:

$$\begin{cases} -x_1 + x_2 + \dots + x_k = w_1 \\ x_1 - x_2 + \dots + x_k = w_2 \\ \dots \\ x_1 + x_2 + \dots - x_k = w_k \end{cases}$$

The solvability (and solutions) of system  $H(w_1, w_2, G)$  can easily be characterized in this case, and depends on whether  $p \nmid (k-2)$  or  $p \mid (k-2)$ . In the former case one can easily check that for any  $w_1, w_2$  system  $H(w_1, w_2, G)$  has an unique solution  $x_i = 2^{-1}[(k-2)^{-1}w - w_i]$ ,  $i = 1, \dots, k$ . In the latter case, the system  $H(w_1, w_2, G)$  has a solution if and only if  $w = 0 \pmod{p}$ . Indeed, the condition follows immediately from summing the equations of the system. On the other hand if  $w = 0$  holds one can easily verify that the following family

$$\begin{cases} x_1 = \lambda \\ x_2 = \lambda + 2^{-1}(w_1 - w_2) \\ x_3 = \lambda + 2^{-1}(w_1 - w_3) \\ \dots \\ x_k = \lambda + 2^{-1}(w_1 - w_k) \end{cases}$$

with  $\lambda$  arbitrary in  $\mathbf{Z}_p$ , represents the family of solutions of system  $H(w_1, w_2, G)$ .

In what follows we will not refer to this case dichotomy, but will simply prove the result by induction over  $x = |H(w_1, w_2, G)|$ . With little risk of ambiguity, we will also denote  $x = (x_1, \dots, x_k)$  a solution of  $H(w_1, w_2, G)$  witnessing the value of the norm.

– **Case  $x = 1$ :**

Let  $i_0$  be the unique index such that  $x_i \neq 0$ . Then  $w_{2,i_0} = w_{1,i_0} - 1$  and  $w_{2,j} = w_{1,j} + 1$  for  $j \neq i_0$ , the equalities being interpreted in  $\mathbf{Z}_p$ . In other words, we need to show how to change state vector  $[w_{1,1}; w_{1,2}; \dots; w_{1,k}]$  into state vector  $[(w_{1,1}+1); (w_{1,2}+1); \dots; (w_{1,i_0}-1); \dots; (w_{1,k}+1)]$ .

If  $w_{1,i_0} \neq 0$  a simple move at  $v_{i_0}$  changes state  $w_1$  into  $w_2$  directly. So the only case that needs a proof is  $w_{1,i_0} = 0$ .

Let  $j \neq i_0$  such that  $w_{1,j} \neq 0$ . Such an index exists since  $w_1 \neq \mathbf{0}$ . Furthermore, by reassigning indices we may assume without loss of generality that  $i_0 = 1$  and  $j = 2$ . Thus target state vector is  $[(p-1); (w_{1,2}+1); \dots; (w_{1,k}+1)]$

1. First, using  $r$  times the trick from Observation 2 at vertices  $v_1$  and  $v_2$  changes  $w_1 = [0; w_{1,2}; \dots; w_{1,k}]$  into  $[0; w_{1,2}; w_{1,3} + r); \dots; (w_{1,k} + r)]$ . We choose  $r \in \{0, 1\} \pmod{p}$  in such a way so that  $w_{1,3} + r \neq 0, (p-1) \pmod{p}$ . Next, apply  $(p-2)$  times the trick in Observation 2 between vertices  $v_2$  and  $v_3$  to turn the state vector into  $[(p-2); w_{1,2}; (w_{1,3} + r); \dots; (w_{1,k} + r - 2)]$ . Apply now a move at  $v_3$  to turn the state vector into  $[(p-1); (w_{1,2}+1); (w_{1,3} + r - 1); \dots; (w_{1,k} + r - 1)]$ .
2. If  $w_{1,2} + 1 \neq (p-1) \pmod{p}$  then by applying  $2-r$  times  $\pmod{p}$  the trick in Observation 2 to vertices  $v_1$  and  $v_2$  we reach the desired final state.

3. Suppose we cannot reach case 2 for **any** choice of  $j$  with  $w_j \neq 0$ . Therefore, vector  $w_1$  contains only zeros and  $(p-1)$ 's, with at least one  $(p-1)$ . Rearranging indices, we may assume  $w_1 = [0; (p-1); 0^{r-1}; (p-1)^{k-r-1}]$ , for some  $1 \leq r \leq k-1$ , and the target vector is  $w_2 = [(p-1); 0; 1^{r-1}; 0^{k-r-1}]$ . This is easy to accomplish: first make  $p-1$  moves at vertex  $v_2$ . Then use Observation 2 twice at vertices  $v_1$  and  $v_2$ , to reach desired state  $w_2$ .
- **Case  $x \geq 2$ :** If either there exist two indices  $i$  with  $x_i \neq 0$ , or only such index exists, but a single move at  $v_i$  moves the configuration to  $w_3 \neq \mathbf{0}$  then we are done: we first make one available move that brings the system to  $w_3$ . Now it is easily checked that system  $H(w_3, w_2, G)$  is solvable and has norm  $x-1$ ; we apply the induction hypothesis.
- The only remaining case is  $w_1 = [1; (p-1); \dots; (p-1)]$  and  $w_2 = [(1-x); (x-1); \dots; (x-1)]$ . This is easily solved: First apply  $2x$  times the trick in Observation 2 to vertices  $v_1$  and  $v_2$  in order to change the state of the system to  $[1; (p-1); (2x-1); \dots; (2x-1)]$ . Then make a move  $p-x \pmod p$  times at  $v_2$ .
- This concludes the proof of the case  $x \geq 2$  and, with it, of case  $m = 1$ .

**Observation 3** If  $w_1, w_2$  are nonzero states differing only on hyperedge  $e$  such that  $w_2$  is reachable from  $w_1$  via moves of edge  $e$  only, then  $w_1$  is reachable in this way from  $w_2$  as well. That is, we can “undo” a sequence of moves on a given edge as long as the initial and the final states are nonzero.

**Proof 3** We can simply reason in the hypergraph  $G_2$  containing edge  $e$  only. Since  $w_2$  is reachable from  $w_1$ , system  $H(w_1, w_2, G_2)$  has a solution  $u$ . It is easy to see that  $-u$  is a solution to  $H(w_2, w_1, G_2)$  and we apply the result proved in Case 1.  $\square$

We can generalize the preceding observation to the case when the hypergraph does not consist of a single edge anymore:

**Observation 4** Let  $P = (e_1, e_2, \dots, e_k)$  be a path in hypergraph  $G$ . Let  $w_1$  be a state such that there exists  $v_1 \in e_1 \setminus e_2$  with  $a = w_1(v_1) \neq 0$ . For  $i = 1, \dots, k-1$  let  $v_{i+1} = e_i \cap e_{i+1}$ ,  $b = w_1(v_k)$ , and assume that  $w_1(v) = 0$



for all  $v \in e_2, e_3, \dots, e_{k-1} \setminus \{v_k\}$ . Then configuration  $w_2$ , specified by

$$w_2(v) = \begin{cases} a - 1, & \text{if } v = v_1, \\ b + 1, & \text{if } v = v_k, \\ w_1(v) + 1, & \text{if } v \in e_1, v \neq v_1, v_2 \\ 0, & \text{if } v \in \{v_2, \dots, v_{k-1}\}, v \neq v_k, \\ 1, & \text{if } v \text{ is another vertex in one of } e_2, \dots, e_{k-1} \\ w_1(v), & \text{otherwise.} \end{cases}$$

is reachable from  $w_1$  (and viceversa) by making moves only along path  $P$ .

Observation 4 informally states that one can “propagate a one” along the path from  $v_1$  to  $v_k$  as long as vertices between the two are initially zero, and then restore the configuration (see Figures 2 and 3).

**Proof 4** The forward moves are easy: schedule, in turn, vertices  $v_1, v_2, \dots, v_{k-1}$ , on edges  $e_1, \dots, e_{k-1}$  respectively. We use the fact that labels of  $v_2, \dots, v_{k-1}$  are initially zero, hence scheduling them in turn increases the label of the next node ( $v_k$ , in case of the last one) by one. The new nodes (except maybe the last) get values equal to one, so they can be scheduled in turn. Vertices that are “internal” to one of the edges  $e_2, \dots, e_{k-1}$  get value 1.

The backward schedule is only a little more complicated: first we “undo” in succession the forward moves on edges  $e_{k-2}, \dots, e_2$ , using Case 1 of the Theorem and the fact that each of  $e_2, \dots, e_{k-1}$  contains at least one “internal” node (whose label is 1). The proof of this last claim crucially uses the last two conditions in the definition of good hypergraphs.

We are left with vertex  $v_2$  with a label of 1. We can use it to restore the correct values on edge  $e_1$  as well.  $\square$

- **Case  $m \geq 2$ :** The proof shares the overall structure with the corresponding proof in the case  $p = 2$  from [1], although the specific cases are not those from [1], but reasonably different ones, employing some combinatorial constructions (e.g. Observation 2) that were not needed in the original proof.

There are two possibilities:

1. **For some  $l \in E$  the system  $H(w_1, w_2, G)$  has a solution  $z$  with  $z_{v,l} = 0$  for all  $v \in l$ .**

Then we define hypergraph  $U = G \setminus \{l\}$ . System  $H(w_1, w_2, U)$  is solvable: indeed, every solution of  $H(w_1, w_2, G)$  such that  $z_{v,l} = 0$  for all  $v \in l$  is a solution of  $H(w_1, w_2, U)$ .

Intuitively we would like to reason inductively, removing edge  $l$  and dealing with hypergraph  $U$  instead of  $G$ . However, removing edge  $l$

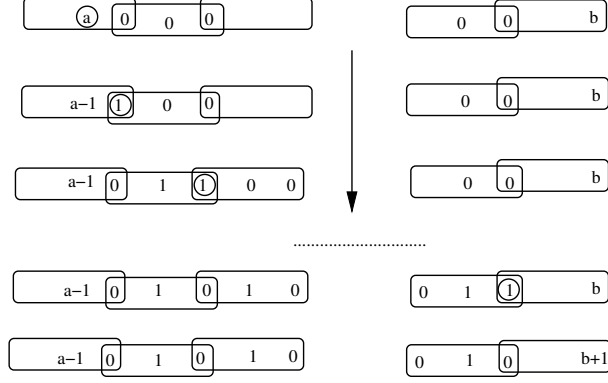


Figure 2: Forward propagation of nonzero values. Scheduled node is circled. Vertex  $v_k$  is last node and edge  $e_k$  is not pictured.

can potentially make hypergraph  $U$  disconnected, and can also create components on which  $w_1$  is identically zero. This creates a problem in changing state from  $w_1$  to  $w_2$  on the connected components of  $U$ . We will show that one can use hyperedge  $l$  to eliminate this problem and "combine" the independent scheduling strategies.

We first classify the components of  $U$  into four categories:

- (a) Connected components  $P$  of  $U$  such that  $w_1|_{P \setminus \{l\}} \neq \mathbf{0}$ .
- (b) Connected components  $Q$  of  $U$  such that  $w_1|_{Q \setminus \{l\}} = \mathbf{0}$  but there exists a vertex  $v \in l \cap Q$  with  $w_1(v) \neq 0$ .
- (c) Connected components  $R$  of  $U$  such that  $w_1|_R \equiv \mathbf{0}$  and  $w_2|_R \neq \mathbf{0}$ .
- (d) Connected components  $S$  of  $U$  such that  $w_1|_S = w_2|_S \equiv \mathbf{0}$ .

Components of type (d) can be eliminated from consideration, as the state on vertices of such components does not change. Also, we may assume that there exists at least one component of type (c), or else we would eliminate such components from consideration as well. Also, since  $H$  was connected, every connected component contains at least one vertex from  $l$ .

**Definition 1** Let  $f : V \rightarrow \{1, 2\}$

$$f(v) = \begin{cases} 1, & \text{if } v \text{ belongs to a component of type (a) or (b),} \\ 2, & \text{if } v \text{ belongs to a component of type (c),} \end{cases}$$

The proof comprises several subcases:

- (a) There exists  $z_1 \in l$  with  $w_1(z_1) \neq 0$  and  $z_2, z_3 \neq z_1$  vertices in  $l$  such that

$$w_{f(z_2)}^{[w_1(z_1, l), z_1, l]}(z_2) \neq w_{f(z_3)}^{[w_1(z_1, l), z_1, l]}(z_3) \quad (2)$$

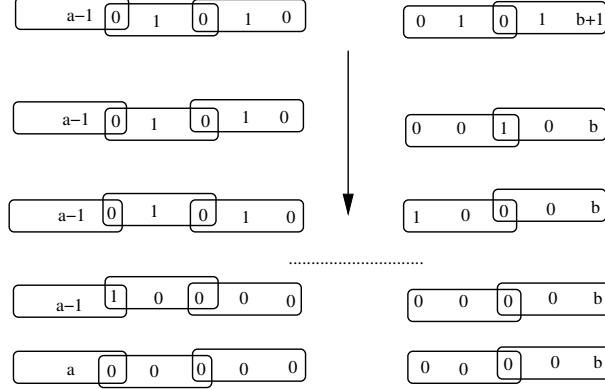


Figure 3: Backward restoration. Each pictured "step" applies Observation 3 on one edge, starting from  $e_{k-1}$  down to  $e_1$

To show that  $w_2$  is reachable from  $w_1$  we employ the following schedule:

- i. first schedule vertex  $z_1$  in edge  $l$  successively ( $w_1(z_1)$  times) until  $w(z_1) = 0$ . This makes vertices  $u \neq z_1$  of  $l$  that belong to components of type (c) assume nonzero values. Indeed, such vertices initially had zero values,  $w_1(z_1)$  was nonzero, and the sum  $w(z_1) + w(u)$  is invariant throughout the process. Also, by definition of connected components of type (c) and the fact that  $w_1(z_1) \neq 0$ , every connected component of this type must contain such a vertex  $u \neq z_1$  of  $l$ . The new state of the system is of course (according to our notation)  $w_3 = w_1^{[w_1(z_1, l), z_1, l]}$ .
- ii. Next we use the induction hypothesis on the connected components  $C$  of type (c) to change the state from  $w_1^{[w_1(z_1, l), z_1, l]}|_C$  to  $w_2^{[w_1(z_1, l), z_1, l]}|_C$ . Here we employ the fact that for every  $k \geq 1$   $w_2^{[k, z_1, l]} - w_1^{[k, z_1, l]} = w_2 - w_1$ , therefore the system associated to the reachability problem on each connected component  $C$  of type (c) is a subset of  $H(w_1, w_2, G)$ , hence solvable. Since vertices  $l$  in  $C$ ,  $l \neq z_1$  have (as we previously argued) nonzero values in  $w_1^{[w_1(z_1, l), z_1, l]}$ , reachability on such components follows by the induction hypothesis.
- iii. Now we apply the trick in Observation 2, scheduling alternatively,  $k_1 = \frac{p-1}{2}$  times, vertices  $z_2$  and  $z_3$ . The crucial aspect is that at the beginning of this process vertices  $z_2$  and  $z_3$  have different values, hence pair  $(z_2, z_3)$  is good at the beginning

of this stage. This is easily seen by the conditions we impose on  $z_2, z_3$ .

The number of alternations is specially chosen so that the outcome of this stage is that labels of all vertices in  $l$ , other than  $z_2, z_3$ , decrease by one. Crucially, at the end of this stage vertex  $z_1$  has label  $p - 1$  (since before this stage its state was zero).

- iv. The next step is to schedule vertex  $z_1$  and edge  $l$   $p - w_1(z_1)$  times. This is possible since  $w_1(z_1) \leq p - 1$ . Crucially, the sequence of moves preserves the property that  $z_2$  and  $z_3$  have different labels, so pair  $(z_2, z_3)$  is good in the resulting state.
- v. Next we apply the trick in Observation 2, scheduling alternately ( $k_2 = \frac{p+1}{2}$  times) vertices  $z_2$  and  $z_3$ .

The number of alternations is chosen so that it increases the state of every vertex in  $l$ , other than  $z_2$  and  $z_3$ , by one. This allows us to compute the state of every vertex  $z \in l$  at this stage:  $w(z) = w_1(z)$  if  $z$  belongs to a component of type (a) or (b),  $w(z) = w_2(z)$  if  $z$  belongs to a component of type (c). Indeed, in the first case the only changes to the state were those imposed by scheduling vertex  $z_1$  a total of  $p$  times (hence the net effect modulo  $p$  is null) and scheduling vertices  $z_2$  and  $z_3$   $p/2$  times each (hence their net effect modulo  $p$  is also null). For vertices in components of type (c) the additional change at step (ii) ensures that at the end of the process their state is  $w(z) = w_2(z)$ .

- vi. Finally, we use the induction hypothesis on  $m$  to change state on components of type (a) and (b) from  $w_1$  to  $w_2$  without employing edge  $l$ . This makes the system assume state  $w_2$  on all vertices.

- (b) There exists  $z_1 \in l$  with  $w_1(z_1) \neq 0$ , subcase (a) does not apply and there exists  $z_4 \notin l$  in a component of type (c) such that  $w_2(z_4) \neq 0$ .

We can reduce this subcase to the previous one as follows: in the previous schedule, before step (iii) use the forward propagation trick in Observation 4 on a shortest path  $P$  from a vertex  $z_4 \notin l$  with  $w_2(z_4) \neq 0$  to some vertex  $z_2$  of edge  $l$ . This changes the state to exactly one vertex of  $l$ , namely  $z_2$ . Moreover  $z_2 \neq z_1$  (since  $z_2$  belongs to a component of type (c)), so the conditions of Subcase (a) are satisfied (with  $z_3$  chosen arbitrarily in  $l \setminus \{z_1, z_2\}$ ). The value of node  $z_2$  does not change throughout the rest of the schedule in Subcase (a), so after its completion we can "undo" the propagation, using Observation 4, from  $z_2$  to  $z_4$ , without employing edge  $l$ .

- (c)  $w_1|_l \neq \mathbf{0}$ , and Subcases (a) and (b) do not apply.

Let  $z_1 \in l$  with  $w_1(z_1) \neq 0$ .

**Claim 1** There exists  $\lambda \neq 0$  such that  $w_{f(x)}(x) = \lambda$  for all  $x \in l$ ,  $x \neq z_1$ .

Moreover, either  $z_1$  is the only vertex of  $l$  in components of type (a) or (b), or the equality is true for  $z_1$  as well.

**Proof 5** Let  $z = Q \cap l$  and  $\lambda = f(z)$ .  $\lambda \neq 0$  since (as Subcase (b) does not apply)  $w_2|_{Q \setminus \{z\}} \equiv 0$  and  $Q$  is a component of type (c). We then apply the fact that Subcase (a) does not apply.

The last part of the claim follows from the fact that no matter how we choose  $z_1$  Subcases (a) and (b) do not apply.  $\square$

We first perform the first two steps of the schedule in subcase (a). The resulting state on edge  $l$  is  $(0; 2\lambda; \dots; 2\lambda)$ . Note that since  $\lambda \neq 0$  and  $p \geq 3$  is prime,  $2\lambda \neq 0$ .

Next, choose an edge  $e$  of  $Q$  that intersects  $l$  at vertex  $v_2$ . Schedule vertex  $v_2$  and edge  $e$ , thus decreasing the label of vertex  $v_2$  by one. This creates a new value on one vertex different from  $z_1$  and allows us to continue with the schedule in Subcase (a). After completing this schedule we "undo" the move at edge  $e$ .

(d)  $w_1|_l \equiv \mathbf{0}$ .

In this case we reduce the problem to one of the previous subcases as follows:  $w_1 \neq \mathbf{0}$ , hence there exist a vertex  $v$  with  $w_1(v) \neq 0$ . Choose such  $v$  at minimal distance from  $l$ , let  $P$  be a path connecting it to a vertex in  $l$ . We first use the "forward propagation trick" in Observation 4 to change the state to state  $w_3$  such that  $w_3|_l \neq \mathbf{0}$ . System  $H(w_3, w_2, G)$  has a solution with  $z_{u,l} = 0$ , for all  $u \in l$ , obtained by changing the solution of  $H(w_1, w_2, G)$  along path  $P$ . We then apply one of the previous cases and conclude that  $w_2$  is reachable from  $w_3$ , hence from  $w_1$ .

2. For all  $l \in E$ , all solutions  $z$  of  $H(w_1, w_2, G)$  satisfy  $z_{v,l} \neq 0$  for some  $v \in l$ .

The proof follows by induction on the norm  $x$  of system  $H(w_1, w_2, G)$ .

- **Case I:**  $x = 1$ . In this case  $w_1$  and  $w_2$  differ on a single edge  $e$ , and we try to convert
- **Case II:**  $x \geq 2$ . Consider a solution  $\bar{z}$  with smallest norm. If for some  $v \in l$  with  $z_{v,l} \neq 0$  scheduling pair  $(v, l)$  once yields a nonzero state  $w_3$  then system  $H(w_3, w_2, G)$  falls into Case 1, or is solvable and has smaller norm. From the induction hypothesis  $w_2$  is reachable from  $w_3$ , hence from  $w_1$ .

We now show that, under the hypothesis that  $m \geq 2$  the remaining alternative cannot happen. Indeed, assuming otherwise, then

for all  $v \in l$  such that  $z_{v,l} \neq 0$ ,  $w_1(v) = 1$ ,  $w_1(u) = p - 1$  for all  $u \neq v \in l$ . Since  $m \geq 2$ , in any two intersecting edges  $e_1, e_2$   $G$  has two vertices  $v_1, v_2$  not in the common intersection with an equal label in  $w_1$  (either 1 or  $p - 1$ ). They cannot be both turned to zero by scheduling a single edge.

□

## 4 From Reachability to Recurrence

We have seen that reachability is easy to test. In the next result we show that recurrence essentially reduces to two reachability tests:

**Theorem 2** *In conditions of Theorem 1, given hypergraph  $G = (E, V)$  and states  $w_1, w_2 \in \mathbf{Z}_p^V$ ,  $w_1 \neq \mathbf{0}$ , state  $w_2$  is a recurrent state for the dynamics started at  $w_1$  if and only if:*

- (1)  $w_2$  is reachable from  $w_1$ .
- (2) State  $\mathbf{0}$  is not reachable from  $w_1$ .

**Proof 6** *Necessity of the two conditions is trivial. Suppose therefore that conditions (1) and (2) are satisfied, and let  $w_3 \in \mathbf{Z}_p^V$  be a state reachable from  $w_1$ .*

*State  $w_3 \neq \mathbf{0}$  because of condition (2). On the other hand let  $Y_1$  be a solution of the system  $H(G, w_1, w_3)$  and  $Y_2$  be a solution of the system  $H(G, w_1, w_2)$ . One can immediately verify that  $Y = Y_2 - Y_1$  (where the difference is taken componentwise in  $\mathbf{Z}_p$ ) is a solution of the system  $H(G, w_3, w_2)$ . Applying Theorem 1 we infer that  $w_2$  is reachable from  $w_3$ .* □

**Corollary 4.1** *Consider the Markov Chain specified by running the  $\mathbf{Z}_p$ -annihilating random walk on a good hypergraph  $G$ .*

1. *Transient states for the dynamics are those states  $\mathbf{0} \neq w \in \mathbf{Z}_p^V$  such that system  $H(G, w, \mathbf{0})$  is solvable.*
2. *All other states are either recurrent or inaccessible, depending on the starting point for the dynamics.*

## 5 Further Comments

It would be interesting (especially in light of Observation 1) to study reachability and recurrence on graphs. Perhaps excluding configuration  $(p-1; p-1; \dots; p-1)$  as a final state (along the lines of the condition imposed in [10] on the final configuration) is enough to recover the connection with linear algebra. Another issue for further study (interesting, in light of the connection with annihilating random walks) is the dynamics of *modular lights-out games under random update*, seen as finite state Markov chains (see [6] Chapter 14 and [1] for related results). Of particular interest is the observation that the antivoter model was used in the analysis of a randomized algorithm for 2-coloring a graph [14]. This was later extended to colorings with more than two colors and/or other coloring restrictions (e.g. [21, 22], see also [23]) and 2-colorings of hypergraphs. Whether cyclic antivoter models and related concepts are useful in the analysis of randomized coloring algorithms is an interesting issue for further study.

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